RESEARCH STATEMENT

DANIEL MINAHAN

I study low–dimensional topology, and in particular the algebraic properties of mapping class groups of surfaces. The mapping class group of a closed surface of genus g, denoted S_g, is the group of homotopy classes of homeomorphisms from S_g to itself, and is denoted $\mathrm{Mod}(S_g)$. Mapping class groups of surfaces appear throughout math in many contexts. For example, $\mathrm{Mod}(S_g)$ appears in algebraic geometry as the orbifold fundamental group of the moduli space of curves $\mathcal{M}_g$. $\mathrm{Mod}(S_g)$ also appears in the study of 3–manifolds via Heegaard splittings, and the cohomology groups of mapping class groups yield characteristic classes for surface bundles. The techniques used to study the mapping class group include tools from algebra, combinatorics, geometry, and topology.

There is a subgroup of $\mathrm{Mod}(S_g)$ given by the kernel of the action of $\mathrm{Mod}(S_g)$ on the first homology $H_1(S_g;\mathbb{Z})$. This group is called the *Torelli group of* S_g and is denoted $\mathcal{I}_g$. Farb and Margalit refer to $\mathcal{I}_g$ as the "non–linear" part of the mapping class group [11, Section 6.6], and Farb says that when one passes to the Torelli group "significant and beautiful new phenomena occur." [10, pg 36].

One of the most significant open questions in the study of mapping class groups is the following, asked by Farb [10, Problem 5.11], Birman [2, Problem 29], Mess [17, Page 90], and Morita [24, Problem 2.1].

Question 1. *For which $g \geq 3$ is $\mathcal{I}_g$ finitely presentable?*

The most direct way to show that $\mathcal{I}_g$ is not finitely presentable would be to show that the homology group $H_2(\mathcal{I}_g;\mathbb{Q})$ is infinite–dimensional. I have the following result [23, Theorem A], which rules out this obstruction. This result also partially answers a question of Bestvina [11, pg. 4] and a folk conjecture (see, e.g., Hain [10, pg. 71] or Putman [27, Conjecture 1.4]).

Theorem A (Minahan)**.** *Let $g \geq 51$. The vector space $H_2(\mathcal{I}_g;\mathbb{Q})$ is finite–dimensional.*

The homology and cohomology of $\mathcal{I}_g$ also appear throughout mathematics in much of the same contexts as the homology and cohomology of $\mathrm{Mod}(S_g)$ or $\mathcal{M}_g$. For instance, elements of $H^k(\mathcal{I}_g;\mathbb{Q})$ yield characteristic classes of surface bundles equipped with markings on the first homology of the fiber. The homology $H_k(\mathcal{I}_g;\mathbb{Q})$ is also the homology of Torelli space, since Torelli space is a $K(\mathcal{I}_g, 1)$.

Many of my results, current projects, and future research directions build off of the approach and techniques that I use to answer Theorem A.

(1) *Other homological properties of the Torelli group.* I will discuss a theorem in progress of mine that describes the rational homology of the Torelli group in a stable range for any degree k, as well as some projects about torsion in the integral homology.

(2) *Connectivity and homotopy types of subcomplexes of the curve complex.* My proof of Theorem A makes use of another one of my results, which says that a certain subcomplex of the curve complex called the complex of homologous curves is highly acyclic. I am interested in extending this result to other subcomplexes of the curve complex, as well as explicitly computing information about the homotopy type of certain other subcomplexes of the curve complex.

(3) *The Johnson filtration.* The Johnson filtration of $\mathrm{Mod}(S_g)$ is a filtration of the mapping class group whose first term is $\mathcal{I}_g$. A result of mine gives information about the top degree homology of the terms of the Johnson filtration. I am also interested in generalizing Theorem A to the terms of the Johnson filtration.

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I will discuss my own contributions in each of these areas, as well as my current projects and future directions I would like to take my research.

In addition, I will discuss some other results of mine not directly related to the Torelli group.

(1) My collaborators and I have explicitly written down formulas for all lines on a cubic surface.
(2) My collaborators and I have shown that the automorphism group of the 1–fine curve graph of a surface is precisely the homeomorphism group of the surface.

Sections 1, 2, and 3 discuss my results about the Torelli group and its subgroups, while Section 4 contains my other results.

1. The Homology of the Torelli group

At present, I have theorems in progress which do the following:

- describe $H_k(\mathcal{I}_g;\mathbb{Q})$ for all $k \geq 2$ and all $g \gg k$, and
- bound or restrict the torsion in $H_k(\mathcal{I}_g;\mathbb{Z})$.

I will also discuss some current projects and open problems of interest related to these two theorems. For technical reasons, we will instead work with $\mathcal{I}_g^1$, which is the Torelli group of the compact, oriented surface S_g^1 of genus g with one boundary component.

Describing $H_k(\mathcal{I}_g^1;\mathbb{Q})$ for $g \gg k$. For each $k \geq 1$, the vector space $H_k(\mathcal{I}_g^1;\mathbb{Q})$ is a representation of the symplectic group $\mathrm{Sp}(2g,\mathbb{Z})$. For any $k \geq 1$, we say that the sequence of vector spaces $\{H_k(\mathcal{I}_g^1;\mathbb{Q})\}_{g\geq 1}$ is *representation stable* if $H_k(\mathcal{I}_g^1;\mathbb{Q})$ admits a description as an $\mathrm{Sp}(2g,\mathbb{Z})$–representation that is independent of g for all $g \gg 0$. The notion of representation stability was introduced by Church–Farb [8] in 2010, and it is now a broad research area. The work of Johnson [16] implies that $\{H_1(\mathcal{I}_g^1;\mathbb{Q})\}_{g\geq 1}$ is representation stable. At present I have proven and am writing up the following theorem, which resolves one of the main conjectures of Church–Farb [8, Conjecture 6.1] in representation stability.

Theorem B (Minahan, in progress)**.** *For all $k \geq 2$, $\{H_k(\mathcal{I}_g^1;\mathbb{Q})\}_{g\geq 1}$ is representation stable.*

As a consequence of a theorem of Kupers–Randal-Williams [18, Section 8.2], Theorem B gives an explicit description of $H_k(\mathcal{I}_g^1;\mathbb{Q})$ for all $k \geq 2$ and $g \gg k$, and also resolves a folk a conjecture (see, e.g., Putman [27, Conjecture 1.4]).

Corollary C (Minahan, given Theorem B)**.** *For any $k \geq 1$ and $g \gg k$, the pushforward map*

$$H_k(\mathcal{I}_g^1;\mathbb{Q}) \to \wedge^k H_1(\mathcal{I}_g^1;\mathbb{Q})$$

is an injection. In particular, $H_k(\mathcal{I}_g^1;\mathbb{Q})$ is finite–dimensional for all $k \geq 2$ and $g \gg k$.

Torsion in $H_k(\mathcal{I}_g^1;\mathbb{Z})$. The integral homology of the Torelli group is a finer invariant then the rational homology, since $H_k(\mathcal{I}_g^1;\mathbb{Z})\otimes\mathbb{Q} = H_k(\mathcal{I}_g^1;\mathbb{Q})$. In particular, any finite order elements in $H_k(\mathcal{I}_g^1;\mathbb{Z})$ are not seen in $H_k(\mathcal{I}_g^1;\mathbb{Q})$. Building on the work of Birman–Craggs [3], Johnson [15] showed that $H_1(\mathcal{I}_g^1;\mathbb{Z})$ is finitely generated. The next natural question to ask when studying $H_k(\mathcal{I}_g^1;\mathbb{Z})$ is the following.

Question 2. *Is $H_2(\mathcal{I}_g^1;\mathbb{Z})$ a finitely generated abelian group for all $g \gg 0$?*

A negative answer to this question for a particular g would imply that $\mathcal{I}_g$ is not finitely presented for that choice of g. My proof of Theorem A uses a result of Putman [26], which says that if $S \hookrightarrow S_g$ is a so–called "clean" embedding of surfaces with genus $g(S) \geq 3$, then the pushforward $H_1(\mathcal{I}(S);\mathbb{Q}) \to H_1(\mathcal{I}_g^1;\mathbb{Q})$ is injective, where $\mathcal{I}(S)$ is the restriction of $\mathcal{I}_g^1$ to the subsurface S. Answering Question 2 requires solving the following problem.

Problem 3. *Show that if $S \hookrightarrow S_g^1$ is a clean embedding of surfaces with $g(S) \geq 3$, then the pushforward $H_1(\mathcal{I}(S);\mathbb{Z}) \to H_1(\mathcal{I}_g^1;\mathbb{Z})$ is injective.*

I have a promising method for solving this problem in the cases relevant for answering Question 2, but there are still details left to be worked out.

Another natural question to ask about $H_k(\mathcal{I}_g^1;\mathbb{Z})$ is the following.

Question 4. *What are orders of the torsion elements in $H_k(\mathcal{I}_g^1;\mathbb{Z})$?*

In the case that $k = 1$, Johnson [15] used the Birman–Craggs–Johnson homomorphism [3] to show that $H_1(\mathcal{I}_g^1;\mathbb{Z})$ has torsion elements, and that all torsion in $H_1(\mathcal{I}_g^1;\mathbb{Z})$ is order 2. Brendle and Farb used the Birman–Craggs–Johnson homomorphism to construct non–trivial torsion classes in $H^2(\mathcal{I}_g^1;\mathbb{Z})$ [4]. A possible method for answering Question 4 would be proving the following analogue of Corollary C.

Problem 5. *Show that for all $k \geq 2$ and $g \gg k$, the pushforward map*

$$H_k(\mathcal{I}_g^1;\mathbb{Z}) \to \wedge^k H_1(\mathcal{I}_g^1;\mathbb{Z});\mathbb{Z})$$

is injective.

At present, this problem is completely open. An answer in either the affirmative or negative would be interesting, the former because it would be a major step towards completely describing $H_k(\mathcal{I}_g^1;\mathbb{Z})$ for $g \gg k$, the latter because it would indicate that $H_k(\mathcal{I}_g^1;\mathbb{Z})$ and $H_k(\mathcal{I}_g^1;\mathbb{Q})$ behave in fundamentally different ways. One essential difficulty is that the representation theory of $\mathrm{Sp}(2g,\mathbb{Z})$ over finite fields is not well understood. Many of the standard tools used to study representations of $\mathrm{Sp}(2g,\mathbb{Z})$, e.g., Margulis superrigidity, are not available.

2. Connectivity and homotopy types of subcomplexes of the curve complex

In this section, I will discuss my Theorem D, which shows that a certain $\mathcal{I}_g$–complex called the *complex of homologous curves*, denoted $\mathcal{C}_{\vec{x}}(S_g)$, is Cohen–Macaulay. This is an essential part of my proof of Theorem A. This complex is constructed by taking $\vec{x} \in H_1(S_g;\mathbb{Z})$ to be a nonzero, primitive element. A k–cell of the complex $\mathcal{C}_{\vec{x}}(S_g)$ is $k+1$ isotopy classes of pairwise disjoint essential simple closed curves $c_0, \ldots, c_k$ such that each c_i represents $\vec{x}$ in $H_1(S_g)$.

Theorem D (Minahan)**.** *Let $g \geq 3$ and $\vec{x} \in H_1(S_g;\mathbb{Z})$. The complex of homologous curves $\mathcal{C}_{\vec{x}}(S_g)$ is homologically Cohen–Macaulay of dimension $g-2$.*

This extends a result proven independently and with different methods by Putman [25] and Hatcher–Margalit [13], which says that $\mathcal{C}_{\vec{x}}(S_g)$ is connected for $g \geq 3$. When $\vec{x} = 0$, the resulting complex is called the *complex of separating curves*, and is denoted $\mathcal{C}_{\mathrm{sep}}(S_g)$. Looijenga has shown that $\mathcal{C}_{\mathrm{sep}}(S_g)$ is $(g-3)$–connected [20]. My proof of Theorem D proceeds by studying the a certain complex of cycles $\mathcal{B}_{\vec{x}}(S_g)$ [1], which was constructed by Bestvina–Bux–Margalit. The complex $\mathcal{B}_{\vec{x}}(S_g)$ is contractible [1] and contains

$\mathcal{C}_{\vec{x}}(S_g)$ as a subcomplex. I use a Morse function on $\mathcal{B}_{\vec{x}}(S_g)$ originally defined by Hatcher–Margalit [13] to show that $\mathcal{C}_{\vec{x}}(S_g)$ is $(g-3)$–acyclic. The proof proceeds by a sequence of spectral sequence arguments, each relating the acyclicity of various subcomplexes of $\mathcal{B}_{\vec{x}}(S_g)$ determined by the Morse function to other complexes where the acyclicity can be computed directly.

My main goal currently regarding the complex of homologous curves is to prove the following.

Problem 6. *Show that $\mathcal{C}_{\vec{x}}(S_g)$ is simply connected for $g \geq 4$.*

Resolving this problem would be a possible first step towards answering Question 1. If Problem 6 has an affirmative answer, then $\mathcal{C}_{\vec{x}}(S_g)$ would be homotopy equivalent to a wedge of $(g-2)$–spheres. The next natural question one might ask is the following.

Problem 7. *Explicitly compute representatives of the spheres in $\mathcal{C}_{\text{sep}}(S_g)$.*

The analogous theorem for the curve complex is a result of Broaddus [5].

3. Homological properties of the Johnson filtration

There is a generalization of the Torelli group called the Johnson filtration. The mapping class group $\text{Mod}(S_g)$ acts on $\pi_1(S_g)$ via outer automorphisms. Let $\pi_1^{(i)}(S_g)$ denote the ith term of the lower central series of $\pi_1(S_g)$. The *ith term of the Johnson filtration* is the kernel of the map

$$\text{Mod}(S_g) \to \text{Out}(\pi_1(S_g)/\pi_1^{(i)}(S_g))$$

and is denoted $\mathcal{N}_i(S_g)$. In particular, $\mathcal{I}_g = \mathcal{N}_i(S_g)$. The *cohomological dimension of a group G* is the maximum n such that there is a G–module M with $H^n(G;M) \neq 0$. This can be viewed as a measure of the size of a group. Bestvina–Bux–Margalit [1] proved that $\text{cd}(\mathcal{I}_g) = 3g-5$ and $\text{cd}(\mathcal{N}_2(S_g)) = 2g-3$. I have proven the following result [22].

Theorem E (Minahan)**.** *Let $g \geq 2$ and $i \geq 3$. Then* $\text{cd}(\mathcal{N}_i(S_g)) = 2g-3$.

The proof adapts a method of Bestvina, Bux, and Margalit. We consider the action of $\mathcal{N}_i(S_g)$ on the aforementioned complex $\mathcal{B}_{\vec{x}}(S_g)$ and use the equivariant homology spectral sequence for this action. The essential difficulty lies in computing the homology of $\mathcal{N}_i(S_g)$ in twisted coefficients, which requires examining a variant of the Birman exact sequence for the Johnson filtration.

I am also interested in extending Theorem A to the terms of the Johnson filtration. A theorem of Church–Ershov–Putman [7] tells us that $\mathcal{N}_i(S_g)$ is finitely generated for any $i \geq 1$ and any $g \geq k$, so in particular $H_1(\mathcal{N}_i(S_g);\mathbb{Q})$ is finite–dimensional for $g \geq k$. Showing that $H_2(\mathcal{N}_i(S_g);\mathbb{Q})$ is finite–dimensional for $g \geq i$ is an long–term goal and requires answering many additional questions about the terms of the Johnson filtration. In order to adapt the strategy used in the proof of Theorem A, the following problems most likely need to be solved.

Problem 8. *Give a general recipe for any $i \geq 3$ and $g \geq 2$ for determining when, for two simple closed curves $a, b \subseteq S_g$, there is a $\varphi \in \mathcal{N}_i(S_g)$ such that $\varphi a = b$.*

The case of $i = 2$ is is answered by a theorem of Church [6]. In this same paper, Church connects the case when $i \geq 3$ to the problem of determining the images of the higher Johnson homomorphisms, which are maps to abelian groups that encode information about the Johnson filtration.

Another problem that would need to be answered to extend Theorem A to the Johnson filtration is the following.

Problem 9. *For any $i \geq 2$, $g \geq 2$ construct an $\mathcal{N}_i(S_g)$–complex $X_{i,g}$ such that:*
- *the quotient space $X_{i,g}/\mathcal{N}_i(S_g)$ has one vertex and*
- *the space $X_{i,g}$ is 2–acyclic.*

In the case that $i = 1$, the complex of homologous curves $C_{\vec{x}}(S_g)$ defined in Section 2 section has the desired properties. Giving a precise answer to Problem 9 most likely requires giving an answer to Problem 8. The last main step in extending Theorem A to the terms of the Johnson filtration is solving the following.

Problem 10. *For any $i \geq 2$, $g \gg i$ and $X_{i,g}$ as in Problem 9, show that $H_2(X_{i,g}/\mathcal{N}_i(S_g); \mathbb{Q})$ is finite–dimensional.*

There are also a host of other problems that would need to be solved in order to extend Theorem A to the Johnson filtration. I have no expectation that Theorem A can be extended to the Johnson filtration in the near future, but the necessary steps are interesting and challenging problems in their own right.

4. Other research

I have also completed two other projects, one in algebraic geometry and one in combinatorial topology.

Lines on cubic surfaces. A theorem of Jordan says that any smooth cubic surface $P \subseteq \mathbb{CP}^3$ contains exactly 27 lines. Let $\mathcal{M}_{2,3}$ denote the parameter space of cubic surfaces in $\mathbb{CP}^3$, and let $\mathcal{M}_{2,3}(\ell)$ denote the parameter space of pairs (P, ℓ) where $P \in \mathcal{M}_{2,3}$ and $\ell \subseteq P$ is a line. Harris observed that the covering map $\mathcal{M}_{2,3}(\ell) \to \mathcal{M}_{2,3}$ could be described as a sequence of Galois covers [12, pg. 718–719]. Farb asked if this observation could be used to explicitly write down formulas for the lines on P [9]. My collaborators and I have proven the following [21].

Theorem F (McKean–Minahan–Zhang)**.** *There are explicit formulas for computing the lines on a cubic surface $P \subseteq \mathbb{CP}^3$ in terms of three initial skew lines $\ell_1, \ell_2, \ell_3 \subseteq P$.*

The main content of the paper is actually writing down all of these formulas. This also computationally resolves the problem of determining the number of lines on a real cubic surface by examining when our formulas yield real numbers.

Homeomorphisms of surfaces. It is a theorem of Ivanov that for $g \geq 2$, the natural inclusion $\mathrm{Mod}^{\pm}(S_g) \to \mathrm{Aut}(\mathcal{C}(S_g))$ is an isomorphism [14]. Here $\mathcal{C}(S_g)$ is the curve complex and $\mathrm{Mod}^{\pm}(S_g)$ is the extended mapping class group, i.e., $\mathrm{Homeo}(S_g)/\mathrm{Homeo}_0(S_g)$. Let $\mathcal{C}^{\dagger}(S_g)$ denote the fine curve complex, where a k–cell is a set $\{c_0, \ldots, c_k\}$ of disjoint, simple, essential, properly embedded circles in S_g. Long, Margalit, Pham, Verberne, and Yao showed that the natural inclusion $\mathrm{Homeo}(S_g) \to \mathrm{Aut}(\mathcal{C}^{\dagger}(S_g))$ is an isomorphism [19]. We let $\mathcal{C}^{\dagger}_1(S_g)$ denote the 1–fine curve complex, where a k–cell is a set $\{c_0, \ldots, c_k\}$ of simple, essential, properly embedded circles in S_g such that $|c_i \cap c_j| \leq 1$ for all $0 \leq i < j \leq k$. My collaborators and I have proven the following.

Theorem G (Booth–Minahan–Shapiro)**.** *The map $\mathrm{Homeo}(S_g) \to \mathrm{Aut}(\mathcal{C}^{\dagger}_1(S_g))$ is an isomorphism for $g \geq 1$.*

We also have the following question, originally asked by Gay.

Question 11. *Can the topology of $\mathrm{Homeo}(S_g)$ be recovered from combinatorial information about $\mathcal{C}^{\dagger}_1(S_g)$?*

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Skiles Classroom Building, Georgia Institute of Technology, Atlanta, GA 30332
Email address: `dminahan6@gatech.edu`